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On Stancu–Mühlbach Operators and Some Connected Problems Concerning Probability Distributions

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New approximation properties concerning Beta and Stancu–Mühlbach operators are given. It is shown that both operators preserve Lipschitz constants. We also give quantitative estimates for the approximation of Bernstein, Szász, and Baskakov operators by Stancu–Mühlbach operators, as well as for the approximation of Gamma operators by Beta operators. By duality, these results may be translated into quantitative estimates for the total variation distance from the Pólya distribution to the binomial, Poisson, and negative binomial distributions.

1. INTRODUCTION

The positive linear polynomial operator defined by

$$P_n^\alpha(f, x) = \sum_{k=0}^{n} f(k/n)\, w_{n,k}(x; \alpha), \qquad n \in \mathbb{N},\ x \in [0, 1],\ \alpha \geqslant 0, \tag{1}$$

where f is a real function on $[0, 1]$ and

$$w_{n,k}(x; \alpha) = \binom{n}{k} \frac{\prod_{i=0}^{k-1} (x + i\alpha) \prod_{j=0}^{n-k-1} (1 - x + j\alpha)}{(1+\alpha)(1+2\alpha)\cdots(1+(n-1)\alpha)},$$

was introduced by Stancu [19, 20], who studied, among other properties, the convergence of $P_n^\alpha f$ to f as $n \to \infty$ and $0 \leqslant \alpha = \alpha(n) \to 0$, also providing bounds for $P_n^\alpha(f, x) - f(x)$ under several differentiability assumptions on f. Further results were given by Mühlbach [15, 16]. A generalization of P_n^α is considered in [2].

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Observe that we can write $P_n^\alpha(f, x) = Ef(n^{-1}U_n^{x,\alpha})$, where E denotes mathematical expectation and $U_n^{x,\alpha}$ is a random variable with the Pólya–Eggenberger distribution of parameters n, x, $1-x$, α [8]. In the case $\alpha = 0$, $U_n^{x,0}$ has the binomial distribution and P_n^0 is actually the Bernstein operator. A unified approach to operators associated with distributions arising from urn models can be seen in [7].

On the other hand, the integral operator L^α, acting on the space of all real measurable bounded functions on $(0, 1)$, defined by

$$L^\alpha(f, x) = \int_0^1 f(\theta)\, h_\alpha^x(\theta)\, d\theta, \qquad \alpha > 0,\ x \in (0, 1), \tag{2}$$

where h_α^x is the density of the beta distribution with parameters x/α, $(1-x)/\alpha$,

$$h_\alpha^x(\theta) = \left\{ B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right) \right\}^{-1} \theta^{x/\alpha - 1}(1-\theta)^{(1-x)/\alpha - 1}, \qquad \theta \in (0, 1), \tag{3}$$

was also considered by Mühlbach, who provided a Voronovskaja-type theorem and some other results [15]. If f is defined on $[0, 1]$ we shall use the convention $L^\alpha(f, i) = f(i)$, $i = 0, 1$, when necessary.

Operators slightly different from (2) were introduced and studied by Lupas [12] who called them "Beta" operators. Khan [9] has considered the case $\alpha = 1/n$ and provided new approximation properties. In particular, he has obtained bounds for $L^\alpha(f, x) - f(x)$ when f is continuous or of bounded variation. Moreover, it is easy to see, by usual methods [5, 11, 21], that for $\alpha > 0$, $x \in (0, 1)$, and $f \in C^1(0, 1)$

$$|L^\alpha(f, x) - f(x)| \leqslant 2\left(\frac{\alpha x(1-x)}{1+\alpha}\right)^{1/2} \omega\left(f'; \left(\frac{\alpha x(1-x)}{1+\alpha}\right)^{1/2}\right). \tag{4}$$

On the other hand, as already remarked by Mühlbach, we have, for $\alpha > 0$, $n \in \mathbb{N}$ and any real function f on $[0, 1]$

$$P_n^\alpha f = L^\alpha(P_n^0 f), \tag{5}$$

and this implies that $P_n^\alpha f \geqslant P_{n+1}^\alpha f$, whenever f is convex (a result previously obtained by Stancu [20] in a different way).

Moreover, from (5) we have, for $x \in (0, 1)$

$$P_n^\alpha(f, x) - L^\alpha(f, x) = \int_0^1 \{P_n^0(f, \theta) - f(\theta)\}\, h_\alpha^x(\theta)\, d\theta. \tag{6}$$

This formula has some consequences which apparently have not been pointed out so far. Thus, if $f \in C[0, 1]$ then $P_n^\alpha(f, x) \to L^\alpha(f, x)$ (as $n \to \infty$)

(this also follows, by the Helly–Bray theorem, from the weak convergence of $n^{-1}U_n^{x,\alpha}$ to the beta distribution with parameters x/α, $(1-x)/\alpha$ [6]). Also, each (local or uniform) bound for $P_n^0(f, x)-f(x)$ can be used to give bounds for $P_n^\alpha(f, x)-L^\alpha(f, x)$. In particular we have the following result: If $f\in C[0, 1]$ has a second derivative f'' measurable and bounded on $(0, 1)$ then, for any $\alpha>0$ and $x\in(0, 1)$

$$\lim_{n\to\infty} n\{P_n^\alpha(f, x)-L^\alpha(f, x)\}=\frac{1}{2}\int_0^1 f''(\theta)\,\theta(1-\theta)\,h_\alpha^x(\theta)\,d\theta.$$

In fact this follows from the classical Voronovskaja's theorem by applying the dominated convergence theorem, since [14]

$$|P_n^0(f, \theta)-f(\theta)|\leqslant \tfrac{3}{4}n^{-1/2}\omega(f'; n^{-1/2})\leqslant \tfrac{3}{4}\,\|f''\|\,n^{-1}.$$

In this paper, we provide further approximation properties for both P_n^α and L^α. The main results are stated in the next section. Proofs and the necessary supporting results are given in Section 3.

2. Main Results

Denote by $\mathrm{Lip}_A\,\mu$ the set of all real valued functions f defined on $[0, 1]$ such that

$$|f(x)-f(y)|\leqslant A\,|x-y|^\mu, \qquad x, y\in[0, 1],$$

where $A>0$ and $\mu\in(0, 1]$.

THEOREM 1. *For $f\in C[0, 1]$, the three following statements are equivalent*:

(a) $f\in\mathrm{Lip}_A\,\mu$.

(b) $P_n^\alpha f\in\mathrm{Lip}_A\,\mu$ *for all* $\alpha>0$ *and* $n=1, 2, \ldots$.

(c) $L^\alpha f\in\mathrm{Lip}_A\,\mu$ *for all* $\alpha>0$.

THEOREM 2. *Let n be a fixed integer and let f be a real bounded function on $[0, 1]$. For $\alpha>0$ and $x\in[0, 1]$*

$$|P_n^\alpha(f, x)-P_n^0(f, x)|\leqslant 2n(n-1)\,\omega_2(f; n^{-1})\,\frac{\alpha x(1-x)}{1+\alpha},$$

where $\omega_2(f;\cdot)$ is the second modulus of continuity of f. Therefore, the convergence (as $\alpha\to 0$) is uniform.

THEOREM 3. *Let m be a fixed integer and let f be a real measurable bounded function on* $(0, \infty)$. *For* $\beta > x > 0$

$$|L^{x/m\beta}(f(\beta u), x/\beta) - G_m(f, x)| \leqslant \|f\| \frac{(5m+7)x}{2\beta},$$

where G_m is the Gamma operator defined by

$$G_m(f, x) = \frac{(m/x)^m}{(m-1)!} \int_0^\infty f(\theta)\, \theta^{m-1} e^{-m\theta/x}\, d\theta.$$

Therefore, the convergence (*as* $\beta \to \infty$) *is uniform on each bounded interval* $(0, a)$.

G_m is called Gamma operator in [10, 11] for instance, but it differs substantially from the "Gamma" operator as defined and studied by Müller and Lupas [13, 17, 18].

THEOREM 4. *Let m be a fixed integer and let f be a real bounded function on* $[0, \infty)$. *For* $n > x > 0$

$$|P_{mn}^{x/mn}(f(nu), x/n) - B_m^*(f, x)| \leqslant \|f\| \frac{5(m+3)x}{2n},$$

where B_m^* is the Baskakov operator defined by

$$B_m^*(f, x) = \sum_{k=0}^\infty f(k/m) \binom{m+k-1}{k} \frac{x^k}{(1+x)^{m+k}}.$$

Therefore, the convergence (*as* $n \to \infty$) *is uniform on each bounded interval* $[0, a]$.

THEOREM 5. *Let m and f be the same as in Theorem* 4 *and suppose that the non-negative parameter α depends upon n in such a way that* $\alpha(n) = o(n^{-1})$ $(n \to \infty)$. *Then, for* $n > x > 0$

$$|P_{mn}^\alpha(f(nu), x/n) - S_m(f, x)| \leqslant 4 \|f\| \frac{x}{n} + 2m^2\omega_2(f; m^{-1}) \frac{\alpha x(n-x)}{1+\alpha},$$

where S_m is the Szász–Mirakyan operator defined by

$$S_m(f, x) = e^{-mx} \sum_{k=0}^\infty f(k/m) \frac{(mx)^k}{k!}.$$

Therefore, the convergence (*as* $n \to \infty$) *is uniform on each bounded interval* $[0, a]$.

Theorems 4 and 5 give accuracy to results early noticed by Stancu [20, 21]. Theorem 5 extends Theorem 3(a) in [3]. Theorems 2–5 and other analogous results (cf. [3, Theorems 2 and 3] and [4, Theorems 5 and 7]) approximate certain Bernstein-type operators by other ones. As far as such operators can be represented as mathematical expectations and the results hold for any real continuous and bounded function (on the corresponding interval), these are equivalent, by duality, to statements of convergence for the underlying probability distributions. Duality works in both directions. Thus, Theorem 3 will be shown by obtaining previously a bound for the total variation distance between the probability measures involved (Lemma 1 below), and Theorem 3(a) in [3], which follows from an estimate of the rapidity of convergence in Poisson theorem, plays a fundamental role in the proofs of Theorems 4 and 5. Conversely, the bounds in Theorems 2, 4, and 5 depend upon f only through its norm, and therefore these results provide quantitative estimates for the total variation distance from the Pólya distribution to the binomial, negative binomial, and Poisson distributions (see also [4]).

COROLLARY. (a) *For n fixed, $\alpha > 0$ and $x \in [0, 1]$*

$$\sum_{k=0}^{n} \left| w_{n,k}(x; \alpha) - \binom{n}{k} x^k (1-x)^{n-k} \right| \leqslant 8n(n-1) \frac{\alpha x(1-x)}{1+\alpha}.$$

(b) *For m fixed and $n > x > 0$*

$$\sum_{k=0}^{\infty} \left| w_{mn,k}\left(\frac{x}{n}; \frac{x}{mn}\right) - \binom{m+k-1}{k} \frac{x^k}{(1+x)^{m+k}} \right| \leqslant \frac{5(m+3)x}{2n}.$$

(c) *For m fixed, $n > x > 0$ and $0 \leqslant \alpha = \alpha(n) = o(n^{-1})$ $(n \to \infty)$*

$$\sum_{k=0}^{\infty} \left| w_{mn,k}\left(\frac{x}{n}; \alpha\right) - e^{-mx} \frac{(mx)^k}{k!} \right| \leqslant \frac{4x}{n} + 8m^2 \frac{\alpha x(n-x)}{1+\alpha}.$$

3. PROOFS

Proof of Theorem 1. In view of [10, Theorem 3], to prove (a) implies (b) it suffices to see that $U_n^{x,\alpha}$ has the splitting property. To this end consider, for $0 < x < y < 1$, a random vector (V_n, R_n) with the discrete distribution

$$P(V_n = j, R_n = k) = \frac{\binom{-x/\alpha}{j}\binom{-(y-x)/\alpha}{k}\binom{-(1-y)/\alpha}{n-j-k}}{\binom{-1/\alpha}{n}},$$

for $j, k = 0, 1, \ldots$ with $j + k \leqslant n$. It is easy to see that V_n (resp. $V_n + R_n$) has the same distribution as $U_n^{x,\alpha}$ (resp. $U_n^{y,\alpha}$). Finally

$$ER_n = E(V_n + R_n) - EV_n = EU_n^{y,\alpha} - EU_n^{x,\alpha} = n(y - x),$$

as desired. Now, for $\alpha > 0$, $P_n^\alpha f \to L^\alpha f$ (as $n \to \infty$) and therefore (b) implies (c). Similarly, since $L^\alpha f \to f$ (as $\alpha \to 0$) we have (c) implies (a).

An alternative proof runs as follows: Khan [9] has shown that (a) is equivalent to (c) with $\alpha = 1/n$, $n = 1, 2, \ldots$, but the same argument works if α is taken as a positive continuous parameter. In view of (5), the proof is finished by using the corresponding property for the Bernstein operator [1].

Proof of Theorem 2. Apply (4) to the Bernstein polynomial $g = P_n^0 f$ taking into account that $\omega(g', \delta) \leqslant \|g''\| \delta$ and $\|g''\| \leqslant n(n-1)\, \omega_2(f; 1/n)$.

Proof of Theorem 3. By a change of variable we obtain, for $\beta > x > 0$

$$L^{x/m\beta}(f(\beta u), x/\beta) = \int_0^\infty f(\theta)\, h_{m,\beta}^x(\theta)\, d\theta,$$

where

$$h_{m,\beta}^x(\theta) = \left\{B\left(m, \frac{m\beta}{x} - m\right)\right\}^{-1} (1/\beta)(\theta/\beta)^{m-1} (1 - \theta/\beta)^{(m\beta/x) - m - 1},$$

for $\theta \in (0, \beta)$ and 0 otherwise. Therefore

$$|L^{x/m\beta}(f(\beta u), x/\beta) - G_m(f, x)| \leqslant \|f\| \int_0^\infty |h_{m,\beta}^x(\theta) - g_m^x(\theta)|\, d\theta,$$

where

$$g_m^x(\theta) = \frac{(m/x)^m}{(m-1)!}\, \theta^{m-1} e^{-m\theta/x}, \qquad \theta > 0,$$

and the proof will be complete as soon as we show that the following lemma holds true.

LEMMA 1. *For* $\beta > x > 0$

$$\int_0^\infty |h_{m,\beta}^x(\theta) - g_m^x(\theta)|\, d\theta \leqslant \frac{(5m+7)x}{2\beta}.$$

Proof of Lemma 1. We have

$$\int_0^\infty |h^x_{m,\beta}(\theta) - g^x_m(\theta)|\,d\theta = \int_0^\beta |h^x_{m,\beta}(\theta) - g^x_m(\theta)|\,d\theta + \int_\beta^\infty g^x_m(\theta)\,d\theta \quad (7)$$

and

$$\int_\beta^\infty g^x_m(\theta)\,d\theta \leqslant \beta^{-1}\int_0^\infty \theta g^x_m(\theta)\,d\theta = x/\beta.$$

To estimate the first term on the right-hand side of (7) note that, for $\theta \in (0, \beta)$

$$|h^x_{m,\beta}(\theta) - g^x_m(\theta)| = \left(\frac{m}{x}\right)^m \frac{\theta^{m-1}}{(m-1)!}\,A(\theta),$$

where

$$\begin{aligned} A(\theta) &= \left| e^{-m\theta/x} - \left(1-\frac{\theta}{\beta}\right)^{(m\beta/x)-m-1} \prod_{i=1}^m \left(1-\frac{ix}{m\beta}\right)\right| \\ &\leqslant e^{-m\theta/x}\left|1-\prod_{i=1}^m\left(1-\frac{ix}{m\beta}\right)\right| \\ &\quad + \left|e^{-m\theta/x} - \left(1-\frac{\theta}{\beta}\right)^{m\beta/x}\right| \prod_{i=1}^m\left(1-\frac{ix}{m\beta}\right) \\ &\quad + \left|\left(1-\frac{\theta}{\beta}\right)^{m\beta/x} - \left(1-\frac{\theta}{\beta}\right)^{(m\beta/x)-m-1}\right| \prod_{i=1}^m\left(1-\frac{ix}{m\beta}\right). \end{aligned} \quad (8)$$

The three terms on the r.h.s. of (8) are respectively bounded by

$$e^{-m\theta/x}\sum_{i=1}^m \frac{ix}{m\beta} = \frac{(m+1)x}{2\beta}\,e^{-m\theta/x},$$

$$\begin{aligned} e^{-m\theta/x}\left|1-e^{m\theta/x}\left(1-\frac{\theta}{\beta}\right)^{m\beta/x}\right| &\leqslant e^{-m\theta/x}\left|1-\left(1-\frac{\theta^2}{\beta^2}\right)^{m\beta/x}\right| \\ &\leqslant \frac{m\theta^2}{x\beta}\,e^{-m\theta/x} \end{aligned}$$

and

$$\begin{aligned} &\left|1-\left(1-\frac{\theta}{\beta}\right)^{m+1}\right| \left(1-\frac{\theta}{\beta}\right)^{m\beta/x-m-1} \prod_{i=1}^m\left(1-\frac{ix}{m\beta}\right) \\ &\quad \leqslant \frac{(m+1)\theta}{\beta}\left(1-\frac{\theta}{\beta}\right)^{(m\beta/x)-m-1}\prod_{i=1}^m\left(1-\frac{ix}{m\beta}\right). \end{aligned}$$

The conclusion follows by using elementary properties concerning the gamma and beta functions.

Proof of Theorem 4. We shall use the notations in the former proof. The relation

$$\binom{m+k-1}{k}\frac{x^k}{(1+x)^{m+k}}=\int_0^{\infty} e^{-m\theta}\frac{(m\theta)^k}{k!}\,g_m^x(\theta)\,d\theta, \qquad k=0,1,\ldots$$

clearly implies that $B_m^* f = G_m(S_m f)$.

On the other hand, by a change of variable, we have, for $n > x > 0$,

$$P_{mn}^{x/mn}(f(nu), x/n)=\int_0^n P_{mn}^0(f(nu), \theta/n)\, h_{m,n}^x(\theta)\,d\theta.$$

Thus, we can write

$$\begin{aligned}
&|P_{mn}^{x/mn}(f(nu), x/n)-B_m^*(f,x)|\\
&\quad\leqslant \int_0^n |P_{mn}^0(f(nu),\theta/n)-S_m(f,\theta)|\,h_{m,n}^x(\theta)\,d\theta\\
&\qquad+\left|\int_0^n S_m(f,\theta)\,h_{m,n}^x(\theta)\,d\theta-\int_0^{\infty} S_m(f,\theta)\,g_m^x(\theta)\,d\theta\right|. \qquad (9)
\end{aligned}$$

Now, by Theorem 3(a) in [3], the first term on the r.h.s. of (9) is bounded by $4\,\|f\|\,x/n$ and the second one (which is just $|L^{x/mn}(f_m(nu), x/n) - G_m(f_m, x)|$, where $f_m = S_m f$) does not exceed, by Theorem 3 above

$$\|S_m f\|\,\frac{(5m+7)x}{2n}\leqslant \|f\|\,\frac{(5m+7)x}{2n}.$$

Proof of Theorem 5. By a change of variable we have, for $n > x > 0$

$$P_{mn}^{\alpha}(f(nu), x/n)=\int_0^n P_{mn}^0(f(nu),\theta/n)\,h_{\alpha}^{x/n}(\theta/n)(1/n)\,d\theta,$$

where $h_\alpha^{x/n}$ is defined in (3), and therefore

$$\begin{aligned}
&|P_{mn}^{\alpha}(f(nu), x/n)-S_m(f,x)|\\
&\quad\leqslant \int_0^n |P_{mn}^0(f(nu),\theta/n)-S_m(f,\theta)|\,h_{\alpha}^{x/n}(\theta/n)(1/n)\,d\theta\\
&\qquad+\left|\int_0^n \{S_m(f,\theta)-S_m(f,x)\}\,h_{\alpha}^{x/n}(\theta/n)(1/n)\,d\theta\right|. \qquad (10)
\end{aligned}$$

Again by Theorem 3(a) in [3] the first term on the r.h.s. of (10) is bounded by $4 \|f\| x/n$. The second one can be easily estimated by usual methods and it does not exceed

$$2\left(\frac{\alpha x(n-x)}{1+\alpha}\right)^{1/2} \omega\left((S_m f)', \left(\frac{\alpha x(n-x)}{1+\alpha}\right)^{1/2}\right).$$

The proof is finished by observing that $\omega((S_m f)', \delta) \leqslant \|(S_m f)''\| \delta$ and $\|(S_m f)''\| \leqslant m^2 \omega_2(f; 1/m)$.

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